Barycentric Bash

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1 Introduction

In the world of bashing geometry problems, coordinate bash, complex number bash, and trig bash are probably the most common tools. However there is another tool which can prove to be useful for certain problems, namely the barycentric bash. The basis of these notes and the formula sheet came from the two articles by Evan Chen and Max Schindler, namely "Barycentric Coordinates in Olympiad Geometry" and "Barycentric Coordinates for the Impatient".

2 Formulas

For the main set of formulas and results, see Appendix B of Barycentric Coordinates in Olympiad Geometry. I add a few useful results here.

Theorem 1. *Let P be on segment XY such that $XP/XY = \lambda$. Then $P = (1-\lambda)X + \lambda Y$.*

The proof of the above theorem is immediate, but it is useful to note as it is easy to accidentally swap X, Y in the formula for P.

Theorem 2. *The lines $\ell_i : u_i x + v_i y + w_i z = 0$ $(i = 1, 2)$ intersect at the vector product of (u_1, v_1, w_1) and (u_2, v_2, w_2), i.e.*

$$\begin{vmatrix} i & j & k \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = (v_1 w_2 - w_1 v_2 : w_1 u_2 - u_1 w_2 : u_1 v_2 - v_1 u_2).$$

This formula can be useful to remember in more complicated problems.

Theorem 3. *The lines $\ell_i : u_i x + v_i y + w_i z = 0$ for $i = 1, 2$ are parallel if and only if*

$$\begin{vmatrix} 1 & 1 & 1 \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = u_1 v_2 - v_1 u_2 + v_1 w_2 - w_1 v_2 + w_1 u_2 - u_1 w_2 = 0$$

The proof of this theorem is two lines are parallel iff they don't intersect, i.e. iff the point of intersection we calculate is not actually a point, and this is iff the coordinates sum to 0.

3 Tips

- Think before you leap: just because the problem statement uses the variables A, B, C doesn't mean you should take that triangle to be the reference point of your bash. An alternative choice may greatly simplify the calculations (for example, if the problem includes exactly one circumcircle, you may want to take the triangle corresponding to this circumcircle to be the reference).
- Double check every equation you write as you go along. When bashing out the problem there is a lot of algebra, and it is quite easy to make errors; an error made at the start of a problem can carry on through the whole thing, rendering the final result incorrect. In such cases, assigning part marks is very difficult, and you probably won't get many. Checking equations as you go will take longer, but the potential time save against errors makes it very worthwhile. Pointers on double checking:
  - Is the point/line/circle equation supposed to by symmetric with respect to B, C (etc.)? If so, then your equation should reflect this
  - If you are intersecting two lines (or circles), think about plugging in the result back into the equations to verify your work
  - If there is an expression that shows up a lot, consider replacing it by a variable in some calculations. Sometimes an equation will factor, and this may make it easier to notice the factorization.
- Given any (x, y, z) with $x+y+z \neq 0$, there exists a unique rescaling of (x, y, z) so that it has sum one. In particular, we can work with coordinates being triples (x, y, z) with $x+y+z \neq 0$, with the understanding that it corresponds to the point where we *normalize* the coordinates. This is helpful when considering homogeneous equations: for example the line and circle equations. However, when working with things like distances, the normalized versions are necessary.
- If you use barycentric coordinates, you don't have to do the entire problem with them, and you should avoid blindly applying them. For example, consider the following problem: let ABC be a triangle with a right angle at C. Find the barycentric coordinates for D, the foot of the perpendicular form C to AB. You could find D by calculating the equation of the line CD and intersecting with AB, or you can use similar triangles to get $AD = b^2/c$ and $DB = a^2/c$, whence $D = (a^2 : b^2 : 0)$ (not normalized). Both approaches work, but the second one is quicker and easier.

4 Example

Example 1 (USAMO 2008 P2). *Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.*

Solution. This problem is just begging for a barycentric solution. We have three midpoints, perpendicular bisectors of the sides, and intersecting lines. The only circle comes at the end, where we will just have to check that the coordinates of F satisfy the equation of the circle for ANP.

Initialize $A=(1,0,0), B=(0,1,0), C=(0,0,1)$, and $|AB|=c, |BC|=a, |CA|=b$. We have $M=(0:1:1), N=(1:0:1), P=(1:1:0)$. Line AM thus has equation $y-z=0$. The perpendicular bisector of AC has equation $b^2(x-z)+y(a^2-c^2)=0$, so intersecting this with line AM gives us $E=\left(1:\frac{b^2}{b^2+c^2-a^2}:\frac{b^2}{b^2+c^2-a^2}\right)$. By symmetry, we get $D=\left(1:\frac{c^2}{b^2+c^2-a^2}:\frac{c^2}{b^2+c^2-a^2}\right)$. It follows that the line BD has equation $\frac{c^2}{b^2+c^2-a^2}x-z=0$, and line CE is $\frac{b^2}{b^2+c^2-a^2}x-y=0$. The intersection of these lines is $F=\left(1:\frac{b^2}{b^2+c^2-a^2}:\frac{c^2}{b^2+c^2-a^2}\right)$.

We have now calculated F, so let's calculate the equation of the circle APN. The general form is $-a^2yz-b^2zx-c^2xy+(ux+vy+wz)(x+y+z)=0$, and plugging in point A gives us $u=0$. Plugging in point N gives $w=b^2/2$, and point P gives $v=c^2/2$. Therefore the equation is

$$-a^2yz-b^2zx-c^2xy+\left(\frac{c^2}{2}y+\frac{b^2}{2}z\right)(x+y+z)=0;$$

to solve the problem it suffices to show that the point F satisfies this equation. This is equivalent to

$$\begin{aligned}
0=&-\frac{a^2b^2c^2}{(b^2+c^2-a^2)^2}-\frac{b^2c^2}{b^2+c^2-a^2}-\frac{b^2c^2}{b^2+c^2-a^2}\\
&+\left(\frac{b^2c^2}{2(b^2+c^2-a^2)}+\frac{b^2c^2}{2(b^2+c^2-a^2)}\right)\left(1+\frac{b^2+c^2}{b^2+c^2-a^2}\right)\\
=&-\frac{a^2b^2c^2}{(b^2+c^2-a^2)^2}+\frac{b^2c^2}{b^2+c^2-a^2}\left(-2+1+\frac{b^2+c^2}{b^2+c^2-a^2}\right)\\
=&0,
\end{aligned}$$

as desired. □

5 Determining when to use Barycentric Coordinates

The suitability of a geometry problem for a barycentric bash will fall on a scale, from very unsuitable to very suitable. Some general things to look out for are:

GOOD for barycentric coordinates:

- Points constructed as ratios of lengths (especially midpoints)
- Points constructed via intersections of lines
- Not many circles, or circles passing through similar sets of points
- Incenter/excenter/centroid of reference triangle

BAD for barycentric coordinates :

- Many circles, circles passing through mostly distinct points (e.g. circumcircles of ABC and DEF)
- Cyclic quadrilaterals
- n-gons for $n\geq 5$
- Orthocenter/Circumcenter of triangles

The suitability of a problem for barycentric coordinates will depend greatly person to person. For people who are great at synthetic geometry, barycentric coordinates will likely be a tool only used occasionally, as synthetic solutions are nearly always quicker and nicer to writeup (if you can come up with them). However, to someone who is not great at synthetic geometry and can parse algebra well, they will be able to use barycentric coordinates to turn some problems into a straightforward structured calculation (like the example above).

Where do you fall? A good exercise to try is the following:

> Find any geometry problem, sketch the diagram, and decide if you think barycentric coordinates are a reasonable idea to try. If so, try to determine what choices to make and start doing the proof with barycentric coordinates; keep going as long as you find it reasonable. Afterwards, try to do it synthetically/with other bashing methods, and see what worked best. Some questions to ask yourself: was it straightforward with the bary coords? Would it work with bary coords but the algebra is way too messy to get it done? Was it doable with bary coords but much easier synthetically?

By repeating this, you can get a good grasp on how often and for what problems these methods will be good for.

6 Problems

1. Prove Ceva's theorem with barycentric coordinates.

2. Prove Stewart's theorem (let ABC be a triangle, D on segment BC. Let $AB = c$, $BD = m$, $DC = n$, $CA = b$, $BC = a = m + n$, and then we have $b^2m + c^2n = a(d^2 + mn)$.)

3. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

4. Let ABC be a non-isosceles right-angled triangle ($\angle A = 90°$) and let M be the midpoint of BC. ω_1 is a circle which passes through B, M and is tangent to AC at X. ω_2 is a circle which passes through C, M and is tangent to AB at Y (X, Y and A are on the same side of BC). Prove that XY passes through the midpoint of arc BC (that does not contain A) of the circumcircle of ABC.

5. Let ABC be a triangle and let D be a point on the segment $BC, D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E. The circle ACD meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

6. Let ABC be a triangle. Suppose that X, Y are points in the plane such that BX, CY are tangent to the circumcircle of ABC, $AB = BX, AC = CY$ and X, Y, A are in the same side of BC. If I be the incenter of ABC prove that $\angle BAC + \angle XIY = 180$.

7. Let ABC be an acute-angled triangle whose inscribed circle touches AB and AC at D and E respectively. Let X and Y be the points of intersection of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE and let Z be the midpoint of BC. Prove that the triangle XYZ is equilateral if and only if $\angle A = 60^\circ$.

8. Given a triangle ABC. Point A_1 is chosen on the ray BA so that segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that lines A_1A_2, B_1B_2, C_1C_2 are parallel.

9. Triangle ABC is inscribed in circle ω. Point P lies on line BC such that line PA is tangent to ω. The bisector of $\angle APB$ meets segments AB and AC at D and E respectively. Segments BE and CD meet at Q. Given that line PQ passes through the center of ω, compute $\angle BAC$.

10. Given triangle ABC. One of its excircles is tangent to the side BC at point A_1 and to the extensions of two other sides. Another excircle is tangent to side AC at point B_1. Segments AA_1 and BB_1 meet at point N. Point P is chosen on the ray AA_1 so that $AP = NA_1$. Prove that P lies on the incircle of triangle ABC.

11. Given a non-isosceles triangle ABC, let D, E, and F denote the midpoints of the sides BC, CA, and AB respectively. The circle BCF and the line BE meet again at P, and the circle ABE and the line AD meet again at Q. Finally, the lines DP and FQ meet at R. Prove that the centroid G of the triangle ABC lies on the circle PQR.

12. Let ABC be a triangle with a right angle at C. The angle bisectors of angles A, B meet BC, CA at A_1, B_1 respectively. Let I be the incenter of ABC and O the circumcenter of CA_1B_1. Prove that $OI \perp AB$.

13. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

14. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

15. In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90^\circ$.

16. The incircle of a non-isosceles triangle ABC with the incenter I touches the side BC at D. Let X be a point on arc BC from circumcircle of triangle ABC such that if E, F are feet of perpendicular from X to BI, CI and M is midpoint of EF, then we have $MB = MC$. Prove that $\angle BAD = \angle CAX$.